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Best Proximity Pair Theorems for Multifunctions with Open Fibres

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Let A and B be non-empty subsets of a normed linear space, and $f: A \to B$ be a single valued function. A solution to the functional equation $fx = x$, $(x \in A)$ will be an element x_o in A such that $fx_o = x_o$ (i.e., such that $d(fx, x) = 0$). In the case of non-existence of a solution to the equation $fx = x$, it is natural to explore the existence of an optimal approximate solution that will fulfill the requirement to some extent. In other words, an element x_o in A should be found in such a way that $d(x_o, fx_o) = Min\{d(x, fx): x \in A\}$. Thus, the crux of finding an optimal approximate solution to the aforesaid equation $fx = x$ boils down to ascertaining a solution to the optimization problem $Min\{d(x, fx): x \in A\}$. But, $d(x, fx) \geqslant d(A, B)$ for all $x \in A$. So, in the case of seeking an optimal approximate solution to the aforesaid equation $fx = x$, it should be contemplated to find an element x_o in A such that $d(x_o, fx_o) = d(A, B)$. Indeed, given a multifunction $T: A \to 2^B$ with open fibres, best proximity pair theorems, furnishing the sufficient conditions for the existence of an element $x_o \in A$ such that $d(x_o, Tx_o) = d(A, B)$, are proved in this paper.

Key Words: best proximity pairs; Kakutani factorizable multifunctions; best approximant; multifunctions with open fibres.

1. INTRODUCTION

A great number of practical problems are formulated as certain types of mathematical equations, such as systems of linear or algebraic equations, ordinary or partial differential equations, or functional equations. So, it is of common interest to solve an operator equation of the form $Sx = 0$ where S is defined on some suitable space. The operator equation $Sx = 0$ may be equivalently expressed as a fixed point equation $Tx = x$ in such a way that a solution to the equation $Tx = x$ contributes to the one for the corresponding equation $Sx = 0$. Thus, the significance of fixed point theory stems from the fact that it furnishes an unified approach and constitutes an important tool in solving equations which are not necessarily linear.

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On the other hand, if the fixed point equation $Tx = x$ does not possess a solution, it is contemplated to resolve the problem of finding an element x in the suitable space such that x is in proximity to Tx in some sense. In fact, the "Best approximation pair theorems" and "Best proximity pair theorems" are pertinent to be explored in this direction. In the setting of a topological vector space E with a continuous seminorm p, if T is a mapping with domain A, then a best approximation theorem provides sufficient conditions that ascertain the existence of an element x_o, known as best approximant, such that

$$d(x_o, Tx_o) = d(Tx_o, A)$$

where $d(X, Y) := \mathrm{Inf}\{p(x-y): x \in X \text{ and } y \in Y\}$ for any non-empty subsets X and Y of the space E. Indeed, a classical best approximation theorem, due to Ky Fan [3], states that if K is a non-empty compact convex subset of a Hausdorff locally convex topological vector space E with a continuous seminorm p and $T: K \to E$ is a single valued continuous map, then there exists an element $x_o \in K$ such that

$$p(x_o - Tx_o) = d(Tx_o, K)$$

Later, this result has been generalized, by Sehgal and Singh [9], to the one for continuous multifunctions. It is marked that they have also proved the following generalization [10] of the result due to Prolla [6].

If K is a non-empty approximately compact convex subset of a normed linear space X, $T: K \to X$ a single valued continuous map with $T(K)$ relatively compact and $g: K \to K$ an affine, continuous and surjective single valued map such that g^{-1} sends compact subsets of K onto compact sets, then there exists an element x_o in K such that

$$d(gx_o, Tx_o) = d(Tx_o, K).$$

In the setting of Hausdorff locally convex topological vector spaces, the authors Vetrivel, Veeramani and Bhattacharyya [12] have established existential theorems that guarantee the existence of a best approximant for continuous Kakutani factorizable multifunctions which unify and generalize the known results on best approximations. Despite the fact that the existence of an approximate solution is ensured by best approximation theorems, a natural question that arises in this direction is whether it is possible to guarantee the existence of an approximate solution that is optimal. In other words, if A and B are non-empty subsets of a normed linear space and $T: A \to B$ is a mapping, the point to be mooted is whether one can find an element x_o in A such that

$$d(x_o, Tx_o) = Min\{d(x, Tx) : x \in A\}$$

An affirmative answer to this poser is provided by best proximity pair theorems which are considered in this paper. A best proximity pair theorem analyzes the conditions under which the optimization problem, namely

$$\min_{x \in A} d(x, Tx)$$

has a solution. Indeed, if T is a multifunction from A to B, then

$$d(x, Tx) \geqslant d(A, B)$$

So, the most optimal solution to the problem of minimizing the real valued function $x \to d(x, Tx)$ over the domain A of the multifunction T will be the one for which the value $d(A, B)$ is attained. In view of this standpoint, best proximity theorems are considered in this paper to expound the conditions that assert the existence of an element x_o such that

$$d(x_o, Tx_o) = d(A, B)$$

The pair (x_o, Tx_o) is called a *best proximity pair* of T. If the mapping under consideration is a self-mapping, it may be observed that a best proximity pair theorem boils down to a fixed point theorem under certain suitable conditions. Because of the fact that

$$d(x, Tx) \geqslant d(Tx, A) \geqslant d(A, B) \qquad \text{for all} \quad x \in A,$$

an element x_o satisfying the conclusion of a best proximity pair theorem is a best approximant but the refinement of the closeness between x_o and its image Tx_o is demanded in the case of best proximity pair theorems. Also, a best proximity pair theorem sheds light in another direction that it evolves as a generalization of the problem, considered by Beer and Pai [1], Sahney and Singh [8], Singer [11] and Xu [13], of exploring the sufficient conditions for the non-emptiness of the set

$$\operatorname{Prox}(A, B) := \{(a, b) \in A \times B : d(a, b) = d(A, B)\}$$

In [7], best proximity pair theorems have been considered for a class of upper semicontinuous multifunctions which are not necessarily convex valued and it has been shown that such theorems subsume the known fixed point theorems for convex valued upper semicontinuous multifunctions. So, best proximity theorems will also serve as a natural generalization of fixed point theorems.

The purpose of the present paper is to elicit best proximity pair theorems for multifunctions with open fibres.

2. PRELIMINARIES

This section covers the preliminary notions and the results that will be required in the sequel to establish the main theorems.

Let X and Y be non-empty sets. The collection of all non-empty subsets of X is denoted by 2^X.

A *multifunction* or *set-valued function* from X to Y is defined to be a function that assigns to each element of X a non-empty subset of Y.

If T is a multifunction from X to Y, then it is designated as $T: X \to 2^Y$, and for every $x \in X$, Tx is called a *value* of T.

For $A \subseteq X$, the *image* of A under T, denoted by $T(A)$, is defined as

$$T(A) := \bigcup_{x \in A} Tx$$

For $B \subseteq Y$, the *preimage* or *inverse image* of B under T, denoted by $T^{-1}(B)$, is defined as

$$T^{-1}(B) := \{x \in X : Tx \cap B \neq \phi\}$$

If $y \in Y$, then $T^{-1}(y)$ is called a *fibre* of T.

In what follows, it will be assumed that X and Y are topological spaces.

A multifunction $T: X \to 2^Y$ is said to be *upper semicontinuous* if for every closed subset C of Y, its inverse image $T^{-1}(C)$ is closed in X.

It is known that if $T: X \to 2^Y$ is an upper semicontinuous multifunction with compact values, then $T(K)$ is compact in Y whenever K is a compact subset of X.

A multifunction $T: X \to 2^Y$ is said to be a compact multifunction if $T(X)$ is contained in a compact subset of Y.

A single valued function g from a topological space X to another topological space Y is said to be proper if $g^{-1}(K)$ is compact in X whenever K is compact in Y. It is remarked that if g is continuous and X is a compact space, then the map g is proper.

Let E be a normed linear space.

A non-empty subset A of E is said to be *approximately compact* if for each $y \in E$ and each sequence $\{x_n\}$ in A satisfying the condition that $d(x_n, y) \to d(y, A)$, there is a subsequence of $\{x_n\}$ converging to an element of A.

If A is a non-empty approximately compact convex subset of E, then the set $P_A(y)$ of all best approximations in A to any element $y \in A$ defined by

$$P_A(y) := \{x \in A : d(y, x) = d(y, A)\}$$

is a non-empty convex compact subset of A and the multifunction $y \to P_A(y)$ is an upper semicontinuous multifunction on E.

A non-empty subset A of E is aid to be *proximinal* if for every $y \in E$, there exists $x \in E$ such that $d(x, y) = d(y, A)$, i.e., if $P_A(y)$ is non-empty for every element $y \in E$.

Let C be a non-empty convex subset of E. A single valued function $g\colon C \to E$ is said to be *quasi affine* if for every real number $r \geqslant 0$ and $x \in E$, the set $\{u \in C\colon d[g(u), x] \leqslant r\}$ is convex.

3. MAIN THEOREMS

This section is devoted to principal results on best proximity pairs.

For any two non-empty subsets A and B of a normed linear space, the following notations are used in the sequel.

$$d(A, B) := \mathrm{Inf}\{d(a, b)\colon a \in A \text{ and } b \in B\}$$

$$\mathrm{Prox}(A, B) := \{(a, b) \in A \times B : d(a, b) = d(a, B)\}$$

$$A_o := \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}$$

$$B_o := \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}$$

If one of the sets A and B is a singleton, say $A = \{x\}$, then $d(A, B)$ is simply written as $d(x, B)$. Also, if $A = \{x\}$ and $B = \{y\}$, then $d(x, y)$ is indicated sometimes to denote $d(A, B)$ which is precisely $\|x - y\|$.

PROPOSITION 3.1. *If A and B are non-empty subsets of a normed linear space E such that $d(A, B) > 0$, then $A_o \subseteq Bd(A)$ and $B_o \subseteq Bd(B)$ where $Bd(X)$ denotes the boundary of X for any $X \subseteq E$.*

Proof. Let x be an arbitrary element of A_o. Then, there exists $y \in B$ such that $d(x, y) = d(A, B)$. Since $d(A, B) > 0$, A and B are disjoint. Let $K := \{(1 - \lambda)\, x + \lambda y : 0 \leqslant \lambda \leqslant 1\}$. Because the convex set K intersects both A and its complement $E - A$, it must intersect the boundary of A. So, there exists $\lambda_o \in [0, 1)$ such that $z := (1 - \lambda_o)\, x + \lambda_o y \in Bd(A)$. To show that x lies in the boundary of A, it suffices to prove that λ_o vanishes.

$$\begin{aligned} \text{If } \lambda_o > 0, \text{ then } d(z, y) &= (1 - \lambda_o)\, d(x, y) \\ &= (1 - \lambda_o)\, d(A, B) \\ &< d(A, B), \qquad \text{which is a contradiction.} \end{aligned}$$

So, x is in the boundary of A. Hence, $A_o \subseteq Bd(A)$. Similarly, it may be proved that B_o is contained in the boundary of B. ∎

This section deals with best proximity pair theorems for multifunctions with open fibres. In the main theorem of this section, the underlying multifunction is required to satisfy the following condition.

"For every open set U in the domain space of the multifunction T under consideration, $\cap\{Tx: x \in U\}$ is convex."

It may be noted that this condition is fulfilled in the following two cases:

(i) T is convex valued

(ii) $Tx \cap Ty$ is convex whenever x and y are distinct elements

(A particular situation of case (ii) is that the images of distinct elements under T are disjoint subsets.)

The following example exhibits that this requirement is weaker than that of convexity of all values of the multifunction T. Also, this example does not fall under case (ii).

$$\text{For } x \in (0, 1), \text{ let } A_x := [0, x] \cup (\mathbf{Q} \cap (x, 1])$$
$$B_x := [0, x] \cup ((\mathbf{R} - \mathbf{Q}) \cap (x, 1])$$

where $\mathbf{R}$ is the set of all real numbers and $\mathbf{Q}$ is the set of all rational numbers.

Let $T: [0, 1] \to 2^{[0, 1]}$ be defined as follows:

$$T(x) = \begin{cases} A_x & \text{if } \ x \in (0, 1) \ \text{ and rational} \\ B_x & \text{if } \ x \in (0, 1) \ \text{ and irrational} \end{cases}$$

$T(0) = \mathbf{Q} \cap (0, 1)$ and $T(1) = [0, 1]$. Then, for $a, b \in (0, 1)$ with $a < b$, the following facts may be verified easily.

If $U_1 = [0, a)$, then $\cap\{Tx: x \in U_1\} = \phi$

If $U_2 = (a, b)$, then $\cap\{Tx: x \in U_2\} = [0, a]$

If $U_3 = (b, 1]$, then $\cap\{Tx: x \in U_3\} = [0, b]$

Since any set $U \subseteq K$ which is relatively open in K can be expressed as a union of sets of the form U_1, U_2 and U_3, it follows that $\cap\{Tx: x \in U\}$ is convex. This proves the claim.

The following notations will be used in the sequel.

$$I_n := \{0, 1, 2, ..., n\}$$

$$\Delta_n := Co\{e_0, e_1, e_2, ..., e_n\} \text{ where } \{e_0, e_1, e_2, ..., e_n\}$$
$$\text{is the canonical basis of } R^{n+1}$$

$$\Delta(I) := Co\{e_i: i \in I\} \text{ for any non-empty subset } I \text{ of } I_n.$$

An important tool in the proof of the main theorem of this section is the following theorem due to Horvath [4].

THEOREM A (Horvath [4]). *Let X be a topological space and $G: 2^{I_n} \to 2^X$ be a multifunction such that for each $I \in 2^{I_n}$, $G(I)$ is a contractible subset of X and $G(I) \subseteq G(I')$ whenever $I, I' \in 2^{I_n}$ and $I \subseteq I'$. Then there exists a single valued continuous function $g: \Delta_n \to X$ such that $g[\Delta(I)] \subseteq G(I)$ for all $I \in 2^{I_n}$.*

The proof of the principal theorem of this section invokes a fixed point theorem, due to Lassonde [5], for Kakutani factorizable multifunctions. Before stating it, the following notions are recalled.

A multifunction $T: X \to 2^Y$ from a topological space X to another topological space Y is said to be a *Kakutani multifunction* [5] if the following conditions are satisfied.

(a) T is upper semicontinuous.

(b) Either Tx is a singleton for each $x \in X$ (in which case Y is required to be a Hausdorff topological vector space) or for each $x \in X$, Tx is a non-empty, compact and convex subset of Y (in which case Y is required to be a convex subset of a Hausdorff topological vector space).

The collection of all Kakutani multifunctions from X to Y is denoted by $\mathscr{K}(X, Y)$.

A multifunction $T: X \to 2^Y$ from a topological space X to another topological space Y is said to be a *Kakutani factorizable multifunction* [5] if it can be expressed as a composition of finitely many Kakutani multifunctions.

The collection of all Kakutani factorizable multifunctions from X to Y is denoted by $\mathscr{K}_{\mathscr{C}}(X, Y)$.

If $T = T_1 T_2 \cdots T_n$ is a Kakutani factorizable multifunction then the functions $T_1, T_2, \ldots, T_n$ are known as the *factors* of T.

It may be noted that a Kakutani factorizable multifunction need not be convex valued even though each of its factors is convex valued.

Besides Theorem A, the following fixed point theorem, due to Lassonde [5], for Kakutani factorizable multifunctions will also be invoked to establish the principal best proximity pair theorem for multifunctions with open fibres.

THEOREM B (Lassonde [5]). *If S is a non-empty convex subset of a Hausdorff locally convex topological vector space, then any compact Kakutani factorizable multifunction $T: S \to 2^S$ (i.e., any compact multifunction in the family $\mathscr{K}_{\mathscr{C}}(S, S)$) has a fixed point.*

The main best proximity pair theorem is the following.

THEOREM 3.2. *Let E be a normed linear space. Let A be a non-empty, approximately compact and convex subset of E and B be a non-empty, closed and convex subset of E such that $Prox(A, B)$ is non-empty and A_o is compact. Suppose that*

(a) *$T: A \to 2^B$ is a multifunction such that for every $x \in A_o$, Tx intersects B_o, and for every $y \in B_o$, the fibre $T^{-1}(y)$ is open.*

(b) *For every open set U in A, the set $\cap\{Tu: u \in U\}$ in convex.*

(c) *$g: A \to A$ is a continuous, proper, quasi affine and surjective single valued map such that $g^{-1}(A_o) \subseteq A_o$.*

Then, there exists an element $x_o \in A_o$ such that

$$d(Tx_o, gx_o) = d(A, B)$$

Proof. As $\{T^{-1}(y): y \in B_o\}$ is an open cover for the compact set A_o, there exists a finite subset $\{y_0, y_1, y_2, ..., y_n\}$ of B_o such that

$$A_o \subseteq \cup\{T^{-1}(y_i): i = 0, 1, 2, ..., n\}.$$

Let $S: 2^{I_n} \to \mathscr{P}(A_o)$ (Here $\mathscr{P}(A_o)$ denotes the set of all subsets of A_o) and $F: 2^{I_n} \to 2^{B_o}$ be defined as follows:

For all $I \in 2^{I_n}$,

$$S(I) = \cap\{T^{-1}(y_i): i \in I\}$$

$$F(I) = \begin{cases} \cap\{T(x) \cap B_o: x \in S(I)\} & \text{if} \quad S(I) \neq \phi \\ B_0 & \text{otherwise} \end{cases}$$

Evidently, if $x \in \cap\{T^{-1}(y_i): i \in I\}$, then $y_i \in Tx$ for all $i \in I$. So, $F(I)$ is non-empty, and it is convex by (b). Further, it is easy to see that $F(I) \subseteq F(I')$ whenever $I, I' \in 2^{I_n}$ and $I \subseteq I'$.

By Theorem A, there exists a single valued continuous function $f: \Delta_n \to B_o$ such that $f[\Delta(I)] \subseteq F(I)$ for all $I \in 2^{I_n}$. Let $\{h_0, h_1, ..., h_n\}$ be a partition of unity on the compact set A_o subordinate to the open covering $\{T^{-1}(y_i): i = 0 \text{ to } n\}$.

Let $h: A_o \to \Delta_n$ be defined by

$$h(x) = (h_o(x), h_1(x), h_2(x), ..., h_n(x)) \qquad \text{for all} \quad x \in A_0.$$

Then, h is continuous. Let $G = g^{-1}P_A: B_o \to A_o$. It shall be proved that G is a Kakutani multifunction.

Since g is quasi affine, G is a convex valued multifunction. In fact, if x_1, $x_2 \in Gx$, then it yields that

$$d[g(x_1), x] = d(x, A) = d[g(x_2), x].$$

But, the quasi affinity of g implies that $\{u \in A : d[g(u), x] \leqslant d(x, A)\}$ is convex. So, it follows that

$$d(g[\lambda x_1 + (1-\lambda) x_2], x) = d(x, A)$$

Therefore, Gx is convex.

It is asserted that G is an upper semicontinuous multifunction. Let C be any closed subset of A_o and $\{x_n\}$ be any sequence in $G^{-1}(C)$ such that $x_n \to x$. Then, it follows that, for each n, $G(x_n) \cap C$ is non-empty. Choose an element y_n in $G(x_n) \cap C$ so that $d(g(y_n), x_n) = d(x_n, A)$. Since $y_n \in G(x_n) \subseteq C$, which is a compact set, $\{y_n\}$ has a convergent subsequence. Without loss of generality it may be assumed that $\{y_n\} \to y \in C$.

$$\text{Now, } d[g(y), x] \leqslant d[g(y), g(y_n)] + d[g(y_n), x_n] + d(x_n, x)$$
$$= d[g(y), g(y_n)] + d(x_n, A) + d(x_n, x)$$

Since g is a continuous function,

$$d[g(y_n), g(y)] \to 0.$$

Also, $d(x_n, A) \to d(x, A)$. So, $d[g(y), x] = d(x, A)$ and hence $g(y) \in P_A(x)$. Therefore, $y \in G(x) \cap C$. This ensures that $x \in G^{-1}(C)$. So, $G^{-1}(C)$ is closed and hence G is upper semicontinuous.

Moreover, Gx is compact as g is a proper single valued map and $P_A(x)$ is compact. Also, G is a compact multifunction because both g^{-1} and P_A send compact sets onto compact sets.

Applying Theorem B to the Kakutani factorizable multifunction hGf: $\Delta_n \to \Delta_n$, there exists an element $s_o \in \Delta_n$ such that $s_o \in hGf(s_o)$. So, $s_o \in h(x_o)$ where $x_o \in A_o$ and $g(x_o) \in P_A f s_o$. But, $h(x_o) \in \Delta[I(x_o)]$ where $I(x_o)$ is the set of all indices i such that h_i does not vanish ast x_o. Therefore, it follows that

$$(fh)(x_o) \in f[\Delta(I(x_o))] \subseteq F[I(x_o)] \subseteq T(x_o)$$

Hence, $f(s_o) = (fh)(x_o) \in T(x_o)$.
As $gx_o \in P_A f s_o$, $d(gx_o, fs_o) = d(A, fs_o)$
So, $d(gx_o, Tx_o) \leqslant d(gx_o, fs_o) = d(A, fs_o)$

But, since $fs_o \in B_o$, $d(A, fs_o) = d(A, B)$
So, $d(gx_o, Tx_o) \leqslant d(A, B)$
Also, it is evident that $d(A, B) \leqslant d(gx_o, Tx_o)$.
Thus, $d(gx_o, Tx_o) = d(A, B)$. This completes the proof of the theorem. ∎

The following example illustrates the preceding theorem.

EXAMPLE 3.3. Let $E = \mathbf{R}^2$ with the Euclidean norm.
Let $A := \{(x, 0): 0 \leqslant x \leqslant 1\}$ and $B := \{(x, y): y \geqslant 1 \text{ and } x \geqslant (1/2)\}$

$$\text{Then, } A_o = \{(x, 0): (1/2) \leqslant x \leqslant 1\}$$

$$B_o = \{(x, 1): (1/2) \leqslant x \leqslant 1\}$$

Let $T: A \to 2^B$ be defined as follows.

$$T(x, 0) = \begin{cases} \left\{\left(\frac{1}{2}, 1\right)\right\} & \text{if } \quad x = 0 \\ \left\{\left(\frac{1}{2}, 1\right), \left(\frac{1}{2}, 1 + \frac{1}{2^n}\right)\right\} & \text{if } \quad x = \frac{1}{2^n} \quad (n = 1, 2, 3, \ldots) \\ \left\{(x, 1) : x \leqslant \frac{1}{2}\right\} & \text{otherwise} \end{cases}$$

Let $g: A \to A$ be defined as

$$g(x, y) = (x^2, 0)$$

It is easy to verify that all the conditions of the theorem are satisfied and $d(g(x, 0), T(x, 0)) = 1 = d(A, B)$ for all $x \geqslant 1/\sqrt{2}$.

If T is convex valued, the preceding theorem yields the following result.

COROLLARY 3.4. *Let E be a normed linear space. Let A be a non-empty, approximately compact and convex subset of E and B be a non-empty, closed and convex subset of E such that $Prox(A, B)$ is non-empty and A_o is compact. Suppose that $T: A \to 2^B$ is a convex valued multifunction with open fibres such that $T(A_o) \subseteq B_o$. Then, there exists an element $x_o \in A_o$ such that*

$$d(Tx_o, x_o) = d(A, B)$$

Since the non-emptiness of Prox(A, B) is guaranteed by the compactness of A and the proximinality of B, the following result is a consequence of Corollary 3.4.

Corollary 3.5. *Let E be a normed linear space. Let A be a non-empty compact convex subset of E and B be a non-empty, closed, convex and proximinal subset of E. If $T: A \to 2^B$ is a convex valued multifunction with open fibres such that $T(A_o) \subseteq B_o$, then there exists an element $x_o \in A_o$ such that $d(Tx_o, x_o) = d(A, B)$.*

The preceding result includes the following special case of a fixed point theorem due to Browder [2].

Corollary 3.6. *Let E be a normed linear space and A be a non-empty compact convex subset of E. If $T: A \to 2^A$ is a convex valued multifunction with open fibres, then there exists $x_o \in A$ such that $x_o \in Tx_o$.*

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